

Strongly continuous and locally equi-continuous semigroups on locally convex spaces

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Abstract

We consider locally equi-continuous strongly continuous semigroups on locally convex spaces (X, τ) . First, we show that if (X, τ) has the property that weak* compact sets of the dual are equi-continuous, then strong continuity of the semigroup is equivalent to weak continuity and local equi-continuity.

Second, we consider locally convex spaces (X, τ) that are also equipped with a ‘suitable’ auxiliary norm. We introduce the set \mathcal{N} of τ continuous semi-norms that are bounded by the norm. If (X, τ) has the property that \mathcal{N} is closed under countable convex combinations, then a number of Banach space results can be generalised in a straightforward way. Importantly, we extend the Hille-Yosida theorem.

We apply the results to the study of transition semigroups of Markov processes on complete separable metric spaces.

1 Introduction

The study of Markov processes on complete separable metric spaces (E, d) naturally leads to transition semigroups on $C_b(E)$ that are not strongly continuous with respect to the norm. Often, these semigroups turn out to be strongly continuous with respect to the weaker locally convex *strict* topology.

This leads to the study of strongly continuous semigroups on locally convex spaces. For equi-continuous semigroups, the theory is developed analogously to the Banach space situation for example in Yosida [26]. When characterising the operators that generate a semigroup, the more general context of locally equi-continuous semigroups introduces new technical challenges. Notably, the integral representation of the resolvent is not necessarily available. To solve this problem, Kōmura, Ōuchi and Dembart [5,10,17] have studied various generalised resolvents. More recently, Albanese and Kühnemund [1] also study asymptotic pseudo resolvents and give a Trotter-Kato approximation result and the Lie-Trotter product formula.

A different approach is used in recent papers where a subclass of locally convex spaces (X, τ) is considered for which the ordinary representation of the resolvent can be obtained. Essentially, these spaces are also equipped with a norm $\|\cdot\|$ such

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$(X, \|\cdot\|)$ is Banach and such that the dual $(X, \tau)'$ is norming for $(X, \|\cdot\|)$. Bi-continuous semigroups have been studied in Kühnemund [13] and Albanese and Mangino [2]. Using this approach, the Hille-Yosida and Trotter Approximation theorems have been proven. Bi-continuity has the drawback, however, that it is a non-topological notion. Kunze [14, 15] studies semigroups of which he assumes that the resolvent can be given in integral form. His notions are topological, and he gives a Hille-Yosida theorem for equi-continuous semigroups.

In this paper, we restrict to two subclasses of locally convex spaces for which we can develop the theory of semigroups that are strongly continuous and locally equi-continuous. We say that a locally convex space (X, τ) is of *type A* if weak* compact subsets of the dual are equi-continuous. In Section 3, we show that semigroups on spaces of type A have the property that they are strongly continuous if and only if they are weakly continuous and locally equi-continuous. We introduce a second type of spaces in Section 4. Let the locally convex space (X, τ) be equipped with an auxiliary norm $\|\cdot\|$ such that norm bounded sets coincide with τ bounded sets. We define \mathcal{N} as the set of τ continuous semi-norms that are bounded by the norm. We say that $(X, \tau, \|\cdot\|)$ is of *type B* if \mathcal{N} is closed under taking countable convex combinations. This property allows the generalisation of a number of results in the Banach space theory. First of all, strong continuity of a semigroup on a space of type B implies the exponential boundedness of the semigroup. Second, in Section 5, we show that the resolvent can be expressed in integral form. Third, in Section 6, we give a straightforward proof of the Hille-Yosida theorem for strongly continuous and locally equi-continuous semigroups.

The strength of spaces of type B and the set \mathcal{N} is that results from the Banach space theory generalise by replacing the norm by semi-norms from \mathcal{N} . Technical difficulties arising from working with the set \mathcal{N} instead of the norm are overcome by the probabilistic techniques of stochastic domination and Chernoff's bound.

In Section 7, we apply the results to the study of Markov transition semigroups. First, we introduce the *strict* topology β on $C_b(E)$, where (E, d) is a complete separable metric space. β has the property that the continuous dual of $(C_b(E), \beta)$ is the set of Radon measures. Furthermore, the results in Wiweger and Sentilles [18, 25], show that $(C_b(E), \beta)$, together with the supremum norm, is of type A and of type B.

Under some natural conditions for Markov transition semigroups, we show that strongly continuous semigroups on $(C_0(E), \|\cdot\|)$ correspond to strongly continuous semigroups on $(C_b(E), \beta)$.

A second result is in the area of the *martingale problem*. Solving the martingale problem is a technique that is used frequently to construct a Markov process corresponding to a given generator. We show, under some compactness conditions, that the solution to a martingale problem yields a semigroup that is strongly continuous on $(C_b(E), \beta)$. As a consequence, we show that the generator of this semigroup is an extension of the generator of the martingale problem.

2 Preliminaries

We start with some notation. Let (X, τ) be a locally convex space. We call the family of operators $\{T(t)\}_{t \geq 0}$ a *semigroup* if $T(0) = 1$ and $T(t)T(s) = T(t+s)$ for $s, t \geq 0$. A family of (X, τ) continuous operators $\{T(t)\}_{t \geq 0}$ is called a

strongly continuous semigroup if $t \mapsto T(t)x$ is continuous and *weakly continuous* if $t \mapsto \langle T(t)x, x' \rangle$ is continuous for every $x \in X$ and $x' \in X'$.

Furthermore, we call $\{T(t) \mid t \geq 0\}$ *locally equi-continuous* family if for every $t \geq 0$ and continuous semi-norm p , there exists a continuous semi-norm q such that $\sup_{s \leq t} p(T(s)x) \leq q(x)$ for every $x \in X$.

Finally, we call $\{T(t) \mid t \geq 0\}$ *quasi equi-continuous* family if there exists $\omega \in \mathbb{R}$ such that for every continuous semi-norm p , there exists a continuous semi-norm q such that $\sup_{s \geq 0} e^{-\omega t} p(T(s)x) \leq q(x)$ for every $x \in X$.

We use the following notation for duals and topologies. X^* is the algebraic dual of X and X' is the continuous dual of (X, τ) . We use the acronyms $(X, \sigma(X, X'))$, $(X, \mu(X, X'))$, $(X, \beta(X, X'))$, for X equipped with the weak, Mackey or strong topology. Similarly, we define the weak, Mackey and strong topologies on X' . Also, X^+ is the sequential dual of X :

$$X^+ := \{f \in X^* \mid f(x_n) \rightarrow 0, \text{ for every sequence } x_n \in X \text{ converging to } 0\}.$$

For a set $S \subseteq X$, we denote the *convex hull* of S and the *absolutely convex hull* of S by

$$\begin{aligned} ch(S) &:= \left\{ \sum_{i=1}^n \alpha_i x_i \mid n \geq 1, x_i \in S, \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1 \right\} \\ ach(S) &:= \left\{ \sum_{i=1}^n \alpha_i x_i \mid n \geq 1, x_i \in S, \sum_{i=1}^n |\alpha_i| = 1 \right\}. \end{aligned}$$

3 Connecting strong continuity and local equi-continuity

We start with a small exposition on a subclass of locally convex spaces that imply nice ‘local’ properties of semigroups.

In the proof of Proposition I.5.3 in Engel and Nagel [6] of local equi-continuity in the case of Banach spaces an appeal is made to the Banach-Steinhaus theorem. To be precise, it uses that the set of operators $\{T(t)\}_{t \geq 0}$ is strongly continuous at $t = 0$ and the Banach Steinhaus theorem.

Clearly, this approach partially disregards the fact that the set of operators under consideration is not just an arbitrary set of operators, but one that is strongly continuous for all $t \geq 0$. We show how to use this extra property, and use it to show local equi-continuity, without the Banach-Steinhaus theorem, for a larger class of spaces than the barrelled spaces.

Condition. A locally convex space (X, τ) is of *type A* if

- (a) (X, τ) is sequentially complete.
- (b) all $\sigma(X', X)$ compact sets in X' are equi-continuous on (X, τ) .

We start with a proposition that indicates when (b) is satisfied.

Proposition 3.1. *If a locally convex space (X, τ) is sequentially complete, then it is of type A if it satisfies either of the following properties*

(a) The space (X, τ) is Mackey and the continuous dual X' of X is equal to the sequential dual X^+ of X .

(b) (X, τ) is bornological.

A space for which $X^+ = X'$ is called a *Mazur space*, see for example Wilansky [24].

Proof. First, we show that (b) implies (a). A sequentially complete bornological space is barrelled, see 28.1.(2) in Köthe [11], and barrelled spaces are Mackey [11, 21.4.(4)]. Furthermore, every sequentially continuous mapping from a bornological space into a locally convex space is continuous by [11, 28.3.(4)]. (a) implies that $(X', \mu(X', X))$ is complete by Corollary 3.6 in Webb [23], that states that if (X, τ) is sequentially complete and has the property that $X' = X^+$, then $(X', \mu(X', X))$ is complete.

Finally, we show that if $(X', \mu(X', X))$ is complete, then (X, τ) is of type A.

Let $K \subseteq X'$ be $\sigma(X', X)$ compact. By Krein's theorem [11, 24.4.(4)], the completeness of $(X', \mu(X', X))$ implies that the absolutely convex cover of K is also $\sigma(X', X)$ compact. By the fact that τ is the Mackey topology, every absolutely convex compact set in $(X', \sigma(X', X))$ is equi-continuous [11, 21.4.(1)]. This implies that K is also equi-continuous. \square

Let (E, d) be a complete separable metric space. In Section 7, where we apply the results to Markov transition operators, we show that $(C_b(E), \beta)$, where β is the *strict* topology, is of type A.

The next lemma is the main technical lemma in this Section and will have two important consequences. See also Lemma 3.8 in Kunze [14], who considered a variant of this lemma.

Lemma 3.2. *If a semigroup $\{T(t)\}_{t \geq 0}$ of continuous operators on a locally convex space (X, τ) of type A is strongly continuous, then the semigroup is locally equi-continuous.*

Proof. Fix $T \geq 0$. It follows from 39.3.(4) in Köthe [12] that $\{T(t)\}_{t \leq T}$ is equi-continuous if the set

$$\mathcal{T}'(U) := \{T'(t)x' \mid t \leq T, x' \in U\}$$

is equi-continuous in X' for every equi-continuous set $U \subseteq X'$. So pick an equi-continuous set U in X' . First of all, note that we can replace U by its $\sigma(X', X)$ closure, because the $\sigma(X', X)$ closure of an equi-continuous set is equi-continuous. We show that $\mathcal{T}'(U)$ is relatively compact, so that the fact that (X, τ) is of type A implies that $\mathcal{T}'(U)$ is equi-continuous.

Pick a net $\alpha \mapsto T'(t_\alpha)\mu_\alpha$, where $t_\alpha \leq T$ and $\mu_\alpha \in U$. The interval $[0, T]$ is compact, and because U is closed and equi-continuous it is $\sigma(X', X)$ compact by the Bourbaki-Alaoglu theorem [11, 20.9.(4)], which implies that we can restrict ourselves to a net α such that $t_\alpha \rightarrow t_0$ for some $t_0 \leq T$ and $\mu_\alpha \rightarrow \mu_0$ weakly, where $\mu_0 \in U$.

We show that $T'(t_\alpha)\mu_\alpha \rightarrow T'(t_0)\mu_0$ weakly. For every $x \in X$, we have

$$\begin{aligned} & |\langle T(t_\alpha)x, \mu_\alpha \rangle - \langle T(t_0)x, \mu_0 \rangle| \\ & \leq |\langle T(t_\alpha)x, \mu_\alpha \rangle - \langle T(t_0)x, \mu_\alpha \rangle| + |\langle T(t_0)x, \mu_\alpha \rangle - \langle T(t_0)x, \mu_0 \rangle| \end{aligned} \quad (3.1)$$

The second term converges to 0, because $\mu_\alpha \rightarrow \mu_0$ in $(X', \sigma(X', X))$ and the first term goes to zero because the set U is equi-continuous and $\{T(t)\}_{t \geq 0}$ is strongly continuous. \square

The first application of the lemma is the next proposition, which states that strong continuity is determined by local properties of the semigroup.

Proposition 3.3. *A semigroup $\{T(t)\}_{t \geq 0}$ of continuous operators on a locally convex space (X, τ) of type A is strongly continuous if and only if the following two statements hold*

- (i) *There is a dense subset $D \subseteq X$ such that $\lim_{t \rightarrow 0} T(t)x = x$ for every $x \in D$.*
- (ii) *$\{T(t)\}_{t \geq 0}$ is locally equi-continuous.*

This result resembles Proposition I.5.3 in Engel and Nagel [6], where there is also a third equivalent statement $\lim_{t \downarrow 0} T(t)x = x$ for every $x \in X$. In this situation, however, this statement seems strictly weaker.

Proof. Suppose that $\{T(t)\}_{t \geq 0}$ is strongly continuous. (i) follows immediately and (ii) follows from Lemma 3.2.

For the converse, suppose that we have (i) and (ii) for the semigroup $\{T(t)\}_{t \geq 0}$. First, we show that $\lim_{t \downarrow 0} T(t)x = x$ for every $x \in X$. Pick some $x \in X$ and let x_α be an approximating net in D and let p be a continuous semi-norm and fix $\varepsilon > 0$. We have

$$p(T(t)x - x) \leq p(T(t)x - T(t)x_\alpha) + p(T(t)x_\alpha - x_\alpha) + p(x_\alpha - x).$$

Choose α big enough such that the first and third term are smaller than $\varepsilon/3$. This can be done independently of t , for t in compact intervals by the local equi-continuity of $\{T(t)\}_{t \geq 0}$. Now let t be small enough such that the middle term is smaller than $\varepsilon/3$.

We now prove the strong continuity of $\{T(t)\}_{t \geq 0}$. The previous result clearly gives us $\lim_{s \downarrow t} T(s)x = T(t)x$ for every $x \in X$, so we are left to show that $\lim_{s \uparrow t} T(s)x = T(t)x$.

For $h > 0$ and $x \in X$, we have $T(t-h)x - T(t)x = T(t-h)(x - T(h)x)$, so the result follows by the right strong continuity the local equi-continuity of the semigroup $\{T(t)\}_{t \geq 0}$. \square

Using local equi-continuity and the result of Proposition 3.3 implies the following useful result. The method for the proof is similar to the proof of Theorem I.5.8 in Engel and Nagel [6].

Theorem 3.4. *Suppose that we have a semigroup of continuous operators $\{T(t)\}_{t \geq 0}$ on a locally convex space (X, τ) of type A. Then the semigroup is strongly continuous if and only if it is weakly continuous and locally equi-continuous.*

Proof. We only need to prove that weak continuity implies strong continuity. By the local equi-continuity, the theorem can be proven using Proposition 3.3. Therefore, we will show that the set

$$B := \left\{ x \in X \mid \tau - \lim_{t \downarrow 0} T(t)x = x \right\} \quad (3.2)$$

is dense in (X, τ) .

For every $x \in X$, $r > 0$, and $n \in \mathbb{N}$, define the Riemann sums

$$x_{r,n} := \frac{1}{r2^n} \sum_{i=0}^{2^n-1} T\left(\frac{ir}{2^n}\right) x.$$

As $t \mapsto T(t)$ is weakly continuous, $t \mapsto \langle T(t)x, x' \rangle$ is uniformly continuous on $[0, r]$, which implies that $x_{r,n}$ is weakly Cauchy. Also, $x_{r,n}$ is in the convex hull of $I(x, r) := \{T(s)x \mid 0 \leq s \leq r\}$ for every n .

Considered with the weak topology, $I(x, r)$ is the image of the compact metrisable set $[0, r]$ under a continuous map. As a consequence, Theorem 0.1 and remark 4.2 of Voigt [22] show that the closed convex hull of $I(x, r)$ is weakly compact. Therefore, the Cauchy sequence $x_{n,r}$ has a weak limit x_r in $\overline{\text{co}}I(x, r) \subseteq X$, which is the Pettis integral $x_r = \frac{1}{r} \int_0^r T(s)x ds$.

For a given x , it is clear by the definition of $\{x_r\}_{r>0}$ that x_r converges weakly to x as $r \downarrow 0$. Thus, the set $D := \{x_r \mid x \in X, r > 0\}$ is weakly dense in X . The next step is to show that $D \subseteq B$, which implies that also B is weakly dense.

A base of continuous semi-norms for the topology τ is given by $p_{\mathfrak{S}}(x) := \sup_{x' \in \mathfrak{S}} |\langle x, x' \rangle|$, where \mathfrak{S} ranges over all (X, τ) equi-continuous subsets of X' , [11, 21.3.(2)].

Let \mathfrak{S} be an arbitrary equi-continuous set. For a fixed $r > 0$, $x_r \in D$, $h < r$,

$$\begin{aligned} p_{\mathfrak{S}}(T(h)x_r - x_r) &= \sup_{x' \in \mathfrak{S}} \left| \frac{1}{r} \int_h^{h+r} \langle T(s)x, x' \rangle ds - \frac{1}{r} \int_0^r \langle T(s)x, x' \rangle ds \right| \\ &= \sup_{x' \in \mathfrak{S}} \left| \frac{1}{r} \int_r^{h+r} \langle T(s)x, x' \rangle ds - \frac{1}{r} \int_0^h \langle T(s)x, x' \rangle ds \right| \\ &= \sup_{x' \in \mathfrak{S}} \left| \frac{1}{r} \int_r^{h+r} \langle x, T'(s)x' \rangle ds - \frac{1}{r} \int_0^h \langle x, T'(s)x' \rangle ds \right| \\ &\leq \sup_{y' \in \mathcal{T}\mathfrak{S}} \frac{2h}{r} |\langle x, y' \rangle| \\ &= \frac{2h}{r} p_{\mathcal{T}\mathfrak{S}}(x), \end{aligned}$$

where $\mathcal{T}\mathfrak{S} := \overline{\text{co}}\{T'(s)x' \mid x' \in \mathfrak{S}, s \leq 2r\}$ is also equi-continuous by the local equi-continuity of $\{T(t)\}_{t \geq 0}$. Thus, $p_{\mathcal{T}\mathfrak{S}}$ is also a τ continuous semi-norm, which implies that $p_{\mathcal{T}\mathfrak{S}}(x) < \infty$. By sending $h \downarrow 0$, we obtain $p_{\mathfrak{S}}(T(h)x_r - x_r) \rightarrow 0$.

This implies that $D \subseteq B$, so B is also weakly dense. As B is a convex set, it follows that the weak closure is equal to the τ closure by Proposition 36.2 in Treves [21], so the τ -closure of B is X . The final result follows by Proposition 3.3. \square

As in the Banach space situation, it would be nice to have some condition that implies that the semigroup, suitably rescaled is globally bounded. We directly run into major restrictions.

Example 3.5. Consider $C_c^\infty(\mathbb{R})$ the space of test functions, equipped with its topology as a countably strict inductive limit of Fréchet spaces. This space

is complete [21, Theorem 13.1], Mackey [21, Propositions 34.4 and 36.6] and $C_c^\infty(\mathbb{R})^+ = C_c^\infty(\mathbb{R})'$ as a consequence of [21, Corollary 13.1.1].

Define the semigroup $\{T(t)\}_{t \geq 0}$ by setting $(T(t)f)(s) = f(t+s)$. This semigroup is strongly continuous, however, even if exponentially rescaled, it can never be globally bounded by 19.4.(4) [11].

So even if (X, τ) is of type A, we can have semigroups that have undesirable properties. This issue is serious. For example, in the above example, formally writing the resolvent corresponding to the semigroup in its integral form, yields a function which is not in $C_c^\infty(\mathbb{R})$. One can work around this problem, see for example the references mentioned in the introduction.

However, motivated by the study of Markov processes, where the resolvent informally corresponds to evaluating the semigroup at an exponential random time, we would like to study a context in which the ordinary integral representation for the resolvent holds.

4 A second set of conditions: A suitable structure of bounded sets

In this section, we shift our attention to another type of locally convex spaces. As a first major consequence, we are able to show in Corollary 4.6 an analogue of the exponential boundedness of a strongly continuous semigroup on a Banach space. This indicates that we may be able to mimic major parts of the Banach space theory.

Suppose that (X, τ) is a locally convex space, and suppose that X can be equipped with a norm $\|\cdot\|$, such that τ is weaker than the norm topology. It follows that bounded sets for the norm are bounded sets for τ . This means that if we have a τ -continuous semi-norm p , then there exists some $M > 0$ such that $\sup_{x: \|x\| \leq 1} p(x) \leq M$. This means that $p(x) \leq M\|x\|$ for every x , i.e. every τ -continuous semi-norm is dominated by a constant times the norm.

Definition 4.1. Let (X, τ) be equipped with a norm $\|\cdot\|$ such that τ is weaker than the norm topology. Denote with \mathcal{N} the τ -continuous semi-norms that satisfy $p(\cdot) \leq \|\cdot\|$. Furthermore, we say that \mathcal{N} is *countably convex* if for any sequence p_n of semi-norms in \mathcal{N} and $\alpha_n \geq 0$ such that $\sum_n \alpha_n = 1$, we have that $p(\cdot) := \sum_n \alpha_n p_n(\cdot) \in \mathcal{N}$.

Condition. A locally convex space (X, τ) also equipped with a norm $\|\cdot\|$, denoted by $(X, \tau, \|\cdot\|)$ is of *type B* if

- (a) (X, τ) is sequentially complete.
- (b) τ is weaker than the norm topology.
- (c) Both topologies have the same bounded sets.
- (d) \mathcal{N} is countably convex.

Remark 4.2. Suppose we have a Banach space $(X, \|\cdot\|)$ equipped with a weaker locally convex topology τ , such that $Y := (X, \tau)'$ is norming for X , i.e. $\|x\| = \sup_{x' \in Y, \|x'\|' \leq 1} |\langle x, x' \rangle|$, where $\|\cdot\|'$ is the restriction of the operator norm on $(X, \|\cdot\|)'$ to Y . In this situation, the norm is continuous with respect to the

strong $\beta(X, (X, \tau)')$ topology. This implies that if γ is the topology generated by the norm, then $\tau \subseteq \gamma \subseteq \beta(X, (X, \tau)')$, so that τ bounded sets are bounded in norm by the Banach-Mackey theorem, see 20.11.(3) in Köthe [11].

Remark 4.3. Suppose that (X, τ) is a locally convex space, and let $\|\cdot\|$ be an auxiliary norm, such that the norm topology is stronger than τ , but such that the norm topology has less bounded sets than τ .

In this case, it is useful to consider the *mixed* topology $\gamma = \gamma(\|\cdot\|, \tau)$, introduced in Wiweger [25]. Under some compatibility conditions, the norm bounded sets equal the γ bounded sets and this topology is such that a sequence x_n converges to x with respect γ if and only if the sequence is bounded for the norm and is converging to x in the τ topology.

The bi-continuous semigroups introduced in Kühnemund [13] are sequentially strongly continuous and sequentially locally equi-continuous semigroups for the mixed topology. Thus, we obtain that if the mixed topology is sequential, bi-continuity coincides with strong continuity and local equi-continuity for the mixed topology.

We start with some conditions on the space such that \mathcal{N} is countably convex. It is easily proven that this property holds if (X, τ) is sequential. Interestingly, the same spaces that are of type A , if equipped with a suitable norm, also turn out to be of type B .

Proposition 4.4. *Let (X, τ) be a sequentially complete locally convex space that is also equipped with some norm $\|\cdot\|$ such that τ is weaker than the norm topology and such that both topologies have the same bounded sets. The set \mathcal{N} is countably convex if either of the following hold*

- (a) (X, τ) is sequential,
- (b) (X, τ) is Mackey and the continuous dual $(X, \tau)'$ of (X, τ) is equal to the sequential dual $(X, \tau)^+$,
- (c) (X, τ) is separable and $((X, \tau)', \sigma((X, \tau)', X))$ is sequentially complete.

Proof. Pick $p_n \in \mathcal{N}$ and $\alpha_n \geq 0$, such that $\sum_n \alpha_n = 1$. Define $p(\cdot) = \sum_n \alpha_n p_n(\cdot)$. First of all, it is clear that p is a semi-norm. Thus, we need to show that p is τ continuous.

Suppose that (X, τ) is sequential. Then it is enough to show sequential continuity of p . In this case the result follows directly from the dominated convergence theorem, as every p_n is continuous and $p_n(\cdot) \leq \|\cdot\|$.

For the proof of (b) and (c), we first give an explicit form for an arbitrary τ continuous semi-norm $q \in \mathcal{N}$.

For every $y \in X$, we can find a $x'_y \in (X, \tau)'$ such that $|\langle x, x'_y \rangle| \leq q(x)$ for all $x \in X$ and $|\langle y, x'_y \rangle| = q(y)$ by the Hahn-Banach theorem, 17.2.(3) in Köthe [11].

This implies that $q(x) = \sup_{y \in X} |\langle x, x'_y \rangle|$.

As q is continuous, $\{x'_y \mid y \in X\}$ is a τ equi-continuous subset of $(X, \tau)'$. Furthermore, if $\|\cdot\|'$ is the restriction of the operator norm inherited from $(X, \|\cdot\|)'$, we see

$$\|x'_y\|' = \sup_{x: \|x\| \leq 1} |\langle x, x'_y \rangle| \leq \sup_{x: \|x\| \leq 1} q(x) \leq 1.$$

A short argument shows that $B := \{x' \in (X, \tau)' \mid \|x'\|' \leq 1\}$ is $\sigma((X, \tau)', X)$ closed. Define \mathfrak{S} to be the $\sigma((X, \tau)', X)$ closure of the absolutely convex hull of $\{x'_y \mid y \in X\}$. It follows that $\mathfrak{S} \subseteq B$ and $q(x) = \sup_{x' \in \mathfrak{S}} |\langle x, x' \rangle|$, so that also \mathfrak{S} is τ equi-continuous.

We proceed with the proof of (b). First of all, by Proposition 4.3 and Corollary 4.5 in Webb [23] $((X, \tau)', \sigma((X, \tau)', X))$ is sequentially complete.

The sequence of semi-norms p_n are all of the type described above. So let \mathfrak{S}_n be the equi-continuous subset of B that corresponds to p_n . Define the set

$$\mathfrak{S} := \left\{ \lim_{n \rightarrow \infty} \sum_{i=1}^n \alpha_i u_i \mid u_i \in \mathfrak{S}_i \right\},$$

and note that all these limits exist as $((X, \tau)', \sigma(X', X))$ is sequentially complete and all $\mathfrak{S}_n \subseteq B$.

To finish the proof of case (b), we prove two statements. The first one is that $p(x) = \sup_{x' \in \mathfrak{S}} |\langle x, x' \rangle|$, the second is that \mathfrak{S} is τ equi-continuous. Together these statements imply that p is τ continuous.

We start with the first statement. For every $x \in X$, there are $x'_n \in \mathfrak{S}_n$ such that $p_n(x) = \langle x, x'_n \rangle$ by construction. Therefore,

$$p(x) = \sum_{n=1}^{\infty} \alpha_n \langle x, x'_n \rangle = \langle x, \sum_{n=1}^{\infty} \alpha_n x'_n \rangle \leq \sup_{x' \in \mathfrak{S}} |\langle x, x' \rangle|.$$

On the other hand,

$$\sup_{y' \in \mathfrak{S}} |\langle x, y' \rangle| = \sup_{\substack{y'_n \in \mathfrak{S}_n \\ n \geq 1}} |\langle x, \sum_{n=1}^{\infty} \alpha_n y'_n \rangle| \leq \sum_{n=1}^{\infty} \alpha_n \sup_{y'_n \in \mathfrak{S}_n} |\langle x, y'_n \rangle| \leq p(x).$$

Combining these statements, we see that $p(x) = \sup_{x' \in \mathfrak{S}} |\langle x, x' \rangle|$.

We prove the equi-continuity of \mathfrak{S} . Consider \mathfrak{S}_n equipped with the restriction of the $\sigma((X, \tau)', X)$ topology. Define the product space $\mathcal{P} := \prod_{n=1}^{\infty} \mathfrak{S}_n$ and equip it with the product topology. As every closed equi-continuous set is $\sigma((X, \tau)', X)$ compact by the Bourbaki-Alaoglu theorem [11, 20.9.(4)], \mathcal{P} is also compact.

Let $\phi : \mathcal{P} \rightarrow \mathfrak{S}$ be the map defined by $\phi(\{x'_n\}_{n \geq 1}) = \sum_{n \geq 1} \alpha_n x'_n$. Clearly, ϕ is surjective. We prove that ϕ is continuous. Let $\beta \mapsto \{x'_{\beta, n}\}_{n \geq 1}$ be a net converging to $\{x'_n\}_{n \geq 1}$ in \mathcal{P} . Fix $\varepsilon > 0$ and $f \in X$. Now let N be large enough such that $\sum_{n > N} \alpha_n < \frac{1}{4\|f\|} \varepsilon$ and pick β_0 such that for every $\beta \geq \beta_0$ we have that $\sum_{n \leq N} |\langle f, x'_{\beta, n} - x'_n \rangle| \leq \frac{1}{2} \varepsilon$. Then, it follows for $\beta \geq \beta_0$ that

$$\begin{aligned} & |\phi(\{x'_{\beta, n}\}_{n \geq 1}) - \phi(\{x'_n\}_{n \geq 1})| \\ & \leq \sum_{n \leq N} \alpha_n |\langle f, x'_{\beta, n} - x'_n \rangle| + \sum_{n > N} \alpha_n |\langle f, x'_{\beta, n} - x'_n \rangle| \\ & \leq \frac{1}{2} \varepsilon + \sum_{n > N} \alpha_n \|f\| \|x'_{\beta, n} - x'_n\|' \\ & \leq \frac{1}{2} \varepsilon + 2\|f\| \frac{1}{4\|f\|} \varepsilon \\ & = \varepsilon, \end{aligned}$$

where we use in line four that all $x'_{\beta,n}$ and x'_n are elements of B . As a consequence, \mathfrak{S} , as the continuous image of a compact set, is $\sigma((X, \tau)', X)$ compact. As (X, τ) is of type A by Proposition 3.1, \mathfrak{S} is equi-continuous, which in turn implies that p is τ continuous.

The proof of (c) follows along the lines of the proof of (b), however, we can not use that a $\sigma((X, \tau)', X)$ compact set is equi-continuous. We replace this by using separability. We adapt the proof of (b).

As (X, τ) is separable, the $\sigma((X, \tau)', X)$ topology restricted to \mathfrak{S}_n is metrisable by 21.3.(4) in [11]. This implies that the product space $\mathcal{P} := \prod_{n=1}^{\infty} \mathfrak{S}_n$ with the product topology \mathcal{T} is metrisable.

By 34.11.(2) in Köthe [12], we obtain that \mathfrak{S} , as the continuous image of a metrisable compact set, is metrisable. The equi-continuity of \mathfrak{S} now follows from corollaries of Kalton's closed graph theorem, see Theorem 2.4 and Theorem 2.6 in Kalton [9] or 34.11.(6) and 34.11.(9) in [12]. \square

The usefulness of \mathcal{N} becomes clear from the next two results. Intuitively, the next lemma tells us that in the study of semigroups on these locally convex spaces the collection \mathcal{N} replaces the role that the norm plays for semigroups on Banach spaces.

Lemma 4.5. *Let $(X, \tau, \|\cdot\|)$ be of type B and $\{T(t)\}_{t \geq 0}$ be a semigroup of continuous operators. Then the following are equivalent.*

- (a) $\{T(t)\}_{t \geq 0}$ is locally equi-continuous.
- (b) For every $t \geq 0$ there exists $M \geq 1$, such that for every $p \in \mathcal{N}$ there exists $q \in \mathcal{N}$ such that for all $x \in X$

$$\sup_{s \leq t} p(T(s)x) \leq Mq(x).$$

Proof. We only need to prove (a) to (b). Fix some $t \geq 0$. The family $\{T(s)\}_{s \leq t}$ is equi-continuous, so it maps τ bounded sets to τ -bounded sets. As the bounded sets are also norm bounded sets by assumption, it follows that there exists a $M \geq 1$ such that $\sup_{s \leq t} \|T(s)\| \leq M$.

Let $p \in \mathcal{N}$ and define the semi-norm $p_s(x) := \frac{1}{M}p(T(s)x)$, which is continuous by the continuity of $T(s)$. As $\{T(s)\}_{s \leq t}$ is equi-continuous, there exists a continuous semi-norm q such that $p_s(x) \leq q(x)$ for every $s \leq t$. This means that $p_{[0,t]}(x) := \sup_{s \leq t} p_s(x)$ is also a continuous semi-norm, for which we have

$$p_{[0,t]}(x) = \frac{1}{M} \sup_{s \leq t} p(T(s)x) \leq \frac{1}{M} \sup_{s \leq t} \|T(s)x\| \leq \|x\|.$$

\square

Corollary 4.6. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. For a locally equi-continuous semigroup $\{T(t)\}_{t \geq 0}$, there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $T \geq 0$ and every $p \in \mathcal{N}$ there is a $q \in \mathcal{N}$ such that for all $x \in X$*

$$\sup_{t \leq T} e^{-\omega t} p(T(t)x) \leq Mq(x).$$

Proof. Pick $M \geq 1$ such that for every $p \in \mathcal{N}$ there exists $q \in \mathcal{N}$ such that

$$\sup_{t \leq 1} p(T(t)x) \leq Mq(x) \quad (4.1)$$

for every $x \in X$. Without loss of generality, we can always choose $q \in \mathcal{N}$ to dominate p . We use this property to construct an increasing sequence of semi-norms in \mathcal{N} .

Fix some $p \in \mathcal{N}$ and define $q_0 \geq p$ such that it satisfies the property in equation (4.1). Inductively, let $q_{n+1} \in \mathcal{N}$ be a semi-norm such that $q_{n+1} \geq q_n$ and $\sup_{t \leq 1} q_{n+1}(T(t)x) \leq Mq_n(x)$. Now let $t \geq 0$. Express $t = s + n$ where $n \in \mathbb{N}$ and $0 \leq s < 1$, then it follows that

$$p(T(t)x) \leq Mq_0(T(n)) \leq \dots \leq M^{n+1}q_n(x) \leq Me^{t \log M} q_n(x).$$

Setting $\omega = \log M$, we obtain $\sup_{t \leq T} e^{-\omega t} p(T(t)x) \leq Mq_{\lceil T \rceil}(x)$ for every $x \in X$. \square

This last result inspires the following definition, which is clearly analogous to the situation for semigroups in Banach spaces.

Definition 4.7. We say that a semigroup on a space $(X, \tau, \|\cdot\|)$ of type B is of type (M, ω) , $M \geq 1$ and $\omega \in \mathbb{R}$, if for every $p \in \mathcal{N}$ and $T \geq 0$ there exists $q \in \mathcal{N}$ such that

$$\sup_{t \leq T} e^{-\omega t} p(T(t)x) \leq Mq(x)$$

for all $x \in X$. We say that it is of type $(M, \omega)^*$ if

$$\sup_{t \geq 0} e^{-\omega t} p(T(t)x) \leq Mq(x).$$

Furthermore, we define the growth bound ω_0 of $\{T(t)\}_{t \geq 0}$ by

$$\omega_0 := \inf \{ \omega \in \mathbb{R} \mid \exists M \geq 1 \text{ such that } \{T(t)\}_{t \geq 0} \text{ is } (M, \omega)\text{-bounded} \}.$$

It follows that if a semigroup is of type (M, ω) for some M and ω , then it is locally equi-continuous. Furthermore, if it is of type $(M, \omega)^*$ it is quasi equi-continuous.

5 Infinitesimal properties of semigroups

We now start with studying the infinitesimal properties of a semigroup. Next to the local equi-continuity which we assumed for all results in previous section, we will now also assume strong continuity.

We directly state the following weak analogue of Proposition 3.3 for later reference.

Lemma 5.1. *Let $\{T(t)\}_{t \geq 0}$ be a locally equi-continuous semigroup on a locally convex space (X, τ) . Then the following are equivalent.*

- (a) $\{T(t)\}_{t \geq 0}$ is strongly continuous.
- (b) There is a dense subset $D \subseteq X$ such that $\lim_{t \downarrow 0} T(t)x = x$ for all $x \in D$.

The *generator* $(A, \mathcal{D}(A))$ of a strongly continuous locally equi-continuous semigroup $\{T(t)\}_{t \geq 0}$ on a locally convex space (X, τ) is the linear operator defined by

$$Ax := \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$$

for x in the set

$$\mathcal{D}(A) := \left\{ x \in X \mid \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

We say that $(A, \mathcal{D}(A))$ is *closed* if $\{(x, Ax) \mid x \in \mathcal{D}(A)\}$ is closed in the product space $X \times X$ with the product topology.

We say that \mathcal{D} is a *core* for $(A, \mathcal{D}(A))$, if the closure of $\{(x, Ax) \mid x \in \mathcal{D}\}$ in the product space is $\{(x, Ax) \mid x \in \mathcal{D}(A)\}$.

The generator $(A, \mathcal{D}(A))$ satisfies the following well known properties. The proofs can be found for example as Propositions 1.2, 1.3 and 1.4 in Kōmura [10].

Lemma 5.2. *Let (X, τ) be a locally convex space. For the generator $(A, \mathcal{D}(A))$ of a strongly continuous locally equi-continuous semigroup $\{T(t)\}_{t \geq 0}$, we have*

- (a) $\mathcal{D}(A)$ is closed and dense in X .
- (b) For $x \in \mathcal{D}(A)$, we have $T(t)x \in \mathcal{D}(A)$ for every $t \geq 0$ and $\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x$.
- (c) For $x \in X$ and $t \geq 0$, we have $\int_0^t T(s)x ds \in \mathcal{D}(A)$.
- (d) For $t \geq 0$, we have

$$\begin{aligned} T(t)x - x &= A \int_0^t T(s)x ds && \text{if } x \in X \\ &= \int_0^t T(s)Ax ds && \text{if } x \in \mathcal{D}(A). \end{aligned}$$

The integral in (d) should be understood as a τ Riemann integral. This is possible due to the strong continuity and the local-equi continuity of the semigroup. Define the *spectrum* of $(A, \mathcal{D}(A))$ by $\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda - A \text{ is not bijective}\}$, the *resolvent set* $\rho(A) = \mathbb{C} \setminus \sigma(A)$, for $\lambda \in \rho(A)$ the *resolvent* $R(\lambda, A) = (\lambda - A)^{-1}$, and the *continuous resolvent set* by

$$\rho_\tau(A) := \{\lambda \in \rho(A) \mid R(\lambda, A) \text{ is } (\tau)\text{-continuous}\}.$$

Remark 5.3. We will not touch on the subject in this paper, but sequential completeness implies that multiplication in the locally convex algebra of operators on X is bounded. This is enough to develop spectral theory, for references see Section 40.5 in Köthe [12].

Proposition 5.4. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous and locally equi-continuous semigroup with growth bound ω_0 .*

- (a) *If $\lambda \in \mathbb{C}$ is such that the improper Riemann-integral*

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x dt$$

exists for every $x \in X$, then $\lambda \in \rho(A)$ and $R(\lambda, A) = R(\lambda)$.

(b) Suppose that the semigroup is of type (M, ω) . We have for every $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda > \omega$ and $x \in X$ that

$$R(\lambda)x := \int_0^\infty e^{-\lambda t} T(t)x \, dt$$

exists as an improper Riemann integral. Furthermore, $\lambda \in \rho(A)$.

(c) If $\operatorname{Re} \lambda > \omega_0$, then $\lambda \in \rho(A)$.

Proof. The proof of the first item is standard. We give the proof of (b) for completeness. Let λ be such that $\operatorname{Re} \lambda > \omega$. First, for every $a > 0$ the integral $R_a(\lambda) := \int_0^a e^{-\lambda t} T(t)x \, dt$ exists as a τ Riemann integral by the local equicontinuity of $\{T(t)\}_{t \geq 0}$ and the sequential completeness of (X, τ) .

The sequence $n \mapsto R_n(\lambda)x$ is a τ Cauchy sequence for every $x \in X$, because for every semi-norm $p \in \mathcal{N}$ and $m > n \in \mathbb{N}$ there exists a semi-norm $q \in \mathcal{N}$ such that

$$\begin{aligned} p(R_m(\lambda)x - R_n(\lambda)x) &\leq p\left(\int_n^m e^{-\lambda t} T(t)x \, dt\right) \\ &\leq p\left(\int_n^m e^{-t(\lambda - \omega)} e^{-\omega t} T(t)x \, dt\right) \\ &\leq Mq(x) \int_n^m e^{-t(\operatorname{Re} \lambda - \omega)} \, dt \\ &\leq M \|x\| \frac{e^{-\lambda m} - e^{-\lambda n}}{\operatorname{Re} \lambda - \omega}. \end{aligned}$$

Therefore, $n \mapsto R_n(\lambda)x$ converges by the sequential completeness of (X, τ) . (c) follows directly from (a) and (b). \square

We have shown that if $\operatorname{Re} \lambda > \omega_0$, then $\lambda \in \rho(A)$. It turns out that $R(\lambda, A)$ and its powers are continuous.

Theorem 5.5. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. Let $\{T(t)\}_{t \geq 0}$ be a strongly continuous locally equi-continuous semigroup of growth bound ω_0 . For $\lambda > \omega_0$, $R(\lambda, A)$ is a continuous linear map, i.e. $\lambda \in \rho_\tau(A)$. Furthermore, if $\{T(t)\}_{t \geq 0}$ is of type (M, ω) , then we have for every semi-norm $p \in \mathcal{N}$ that there exists a semi-norm $q \in \mathcal{N}$ such that*

$$p(R(\lambda)x) \leq \frac{M}{\operatorname{Re} \lambda - \omega} q(x).$$

Moreover, for every $\lambda_0 > \omega$ and every semi-norm $p \in \mathcal{N}$, there exists a semi-norm $q \in \mathcal{N}$ such that

$$\sup_{\operatorname{Re} \lambda \geq \lambda_0} \sup_{n \geq 0} (\operatorname{Re} \lambda - \omega)^n p((nR(n\lambda))^n x) \leq Mq(x)$$

for every $x \in X$. If $\{T(t)\}_{t \geq 0}$ is of type $(M, \omega)^*$, then the last statement can be strengthened to

$$\sup_{\operatorname{Re} \lambda > \omega} \sup_{n \geq 0} (\operatorname{Re} \lambda - \omega)^n p((nR(n\lambda))^n x) \leq Mq(x).$$

For the proof of the theorem, we will make use of Chernoff's bound and the probabilistic concept of stochastic domination. A short explanation and some basic results are given in Appendix 8.

Proof. In the proof, we will write $\lceil s \rceil$ for the smallest integer $n \geq s$.

We start with the first statement. Let $\omega_0 < \lambda$, and let ω be such that $\omega_0 < \omega < \lambda$. Suppose that the semigroup is of type (M, ω) .

By the local equi-continuity of $\{T(t)\}_{t \geq 0}$, we can find for every semi-norm $p \in \mathcal{N}$ semi-norms $q_n \in \mathcal{N}$, increasing in n , such that $\sup_{s \leq n} e^{-s\omega} p(T(s)x) \leq M q_n(x)$. It follows that

$$\begin{aligned} p\left(\int_0^\infty e^{-\lambda s} T(s)x ds\right) &\leq \int_0^\infty e^{-\operatorname{Re} \lambda s} p(T(s)x) ds \\ &\leq M \int_0^\infty e^{s(\omega - \operatorname{Re} \lambda)} q_{\lceil s \rceil}(x) ds \end{aligned}$$

$q := \int_0^\infty (\operatorname{Re} \lambda - \omega) e^{s(\omega - \operatorname{Re} \lambda)} q_{\lceil s \rceil} ds$ is a continuous semi-norm by the countable convexity of \mathcal{N} and the fact that we integrate over a probability distribution. So indeed $R(\lambda, A)$ is a continuous linear map and

$$p\left(\int_0^\infty e^{-\lambda s} T(s)x ds\right) \leq \frac{M}{\operatorname{Re} \lambda - \omega} q(x).$$

To lighten the notation, note that this result can also be obtained from the result for a $(M, 0)$ bounded semigroup and a suitable rescaling procedure. For the proof of the second statement, we will work with the rescaled semigroup. By iterating the definition of the resolvent, we find

$$(n \operatorname{Re} \lambda R(n\lambda))^n x = \int_0^\infty \frac{(n \operatorname{Re} \lambda)^n s^{n-1}}{(n-1)!} e^{-n\lambda s} T(s)x ds.$$

For a fixed semi-norm $p \in \mathcal{N}$, we can find, as above, a sequence of semi-norms $q_n \in \mathcal{N}$, increasing in n , and independent of λ such that

$$p((n \operatorname{Re} \lambda R(n\lambda))^n x) \leq M \int_0^\infty \frac{(n \operatorname{Re} \lambda)^n s^{n-1}}{(n-1)!} e^{-sn \operatorname{Re} \lambda} q_{\lceil s \rceil}(x) ds$$

for every $x \in X$. On the right hand side, we see a semi-norm

$$q_{n, \operatorname{Re} \lambda} := \int_0^\infty \frac{(n \operatorname{Re} \lambda)^n s^{n-1}}{(n-1)!} e^{-sn \operatorname{Re} \lambda} q_{\lceil s \rceil} ds$$

in \mathcal{N} by the countable convexity of \mathcal{N} and the fact that we integrate with respect to a probability measure. We denote this probability measure by

$$\mu_{n, \operatorname{Re} \lambda}(ds) = \frac{(n \operatorname{Re} \lambda)^n s^{n-1}}{(n-1)!} e^{-sn \operatorname{Re} \lambda} ds,$$

and with $Z_{n, \operatorname{Re} \lambda}$ the random variable with this distribution. As a consequence, we have the following equivalently definitions:

$$q_{n, \operatorname{Re} \lambda} = \int_0^\infty q_{\lceil s \rceil} \mu_{n, \operatorname{Re} \lambda}(ds) = \mathbb{E} [q_{\lceil Z_{n, \operatorname{Re} \lambda} \rceil}].$$

To show equi-continuity of $(n\operatorname{Re}\lambda)^n R(\lambda n)^n$, we need to find one semi-norm q that dominates all $q_{n,\operatorname{Re}\lambda}$ for $n \geq 0$ and $\operatorname{Re}\lambda \geq \lambda_0$. Because $s \mapsto q_{\lceil s \rceil}(x)$ is an increasing and bounded function for every $x \in X$, the result follows by lemma 8.2, if we can find a random variable Y that stochastically dominates all $Z_{n,\operatorname{Re}\lambda}$. If we recall the definition of stochastic domination, this implies that we need to find a random variable that dominates the tail of the distribution of all $Z_{n,\operatorname{Re}\lambda}$. To study the tails, we will use Chernoff's bound.

Let $g(s, \alpha, \beta) := \frac{\beta^\alpha s^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta s}$, $s \geq 0$, $\alpha, \beta > 0$ be the density with respect to the Lebesgue measure of a $\text{Gamma}(\alpha, \beta)$ random variable. Thus, we see that $Z_{n,\operatorname{Re}\lambda}$ has a $\text{Gamma}(n, n\operatorname{Re}\lambda)$ distribution. A $\text{Gamma}(n, n\operatorname{Re}\lambda)$ random variable, can be obtained as the n -fold convolution of $\text{Gamma}(1, n\operatorname{Re}\lambda)$ random variables, i.e. exponential random variables with parameter $n\operatorname{Re}\lambda$. Probabilistically, this means that a $\text{Gamma}(n, n\operatorname{Re}\lambda)$ can be written as the sum of n independent exponential random variables with parameter $n\operatorname{Re}\lambda$. An exponential random variable η that is $\text{Exp}(\beta)$ distributed has the property that $\frac{1}{n}\eta$ is $\text{Exp}(n\beta)$ distributed. Therefore, we obtain that $Z_{n,\operatorname{Re}\lambda} = \frac{1}{n} \sum_{i=1}^n X_{i,\operatorname{Re}\lambda}$ where $\{X_{i,\beta}\}_{i \geq 1}$ are independent copies of an $\text{Exp}(\beta)$ random variable X_β .

This implies that we are in a position to use a Chernoff bound, Proposition 8.4, to control the tail probabilities of the $Z_{n,\operatorname{Re}\lambda}$. An elementary calculation shows that for $0 < \theta < (\operatorname{Re}\lambda)$, we have $\mathbb{E}[e^{\theta X_{\operatorname{Re}\lambda}}] = \frac{\operatorname{Re}\lambda}{\operatorname{Re}\lambda - \theta}$. Evaluating the infimum in Chernoff's bound yields for $c \geq (\operatorname{Re}\lambda)^{-1}$ that

$$\mathbb{P}[Z_{n,\operatorname{Re}\lambda} > c] < e^{-n(c\operatorname{Re}\lambda - 1 - \log c\operatorname{Re}\lambda)}.$$

Define the non-negative function

$$\begin{aligned} \phi : [\lambda_0^{-1}, \infty) \times [\lambda_0, \infty) &\rightarrow [0, \infty) \\ (c, \alpha) &\mapsto c\alpha - 1 - \log c\alpha \end{aligned}$$

so that for $c \geq \lambda_0^{-1}$ and λ such that $\operatorname{Re}\lambda \geq \lambda_0$ we have

$$\mathbb{P}[Z_{n,\operatorname{Re}\lambda} > c] < e^{-n\phi(c, \operatorname{Re}\lambda)}. \quad (5.1)$$

We use this result to find a random variable that stochastically dominates all $Z_{n,\operatorname{Re}\lambda}$ for $n \in \mathbb{N}$ and $\operatorname{Re}\lambda \geq \lambda_0$. Define the random variable Y on $[\lambda_0^{-1}, \infty)$ by setting $\mathbb{P}[Y > c] = \exp\{-\phi(c, \lambda_0)\}$.

First note that for fixed $c \geq \lambda_0^{-1}$, the function $\alpha \mapsto \phi(c, \alpha)$ is increasing. Also note that $\phi \geq 0$. Therefore, it follows by equation (5.1) that for λ such that $\operatorname{Re}\lambda \geq \lambda_0$ and $c \geq \lambda_0^{-1}$, we have

$$\mathbb{P}[Z_{n,\operatorname{Re}\lambda} > c] < e^{-n\phi(c, \operatorname{Re}\lambda)} \leq e^{-n\phi(c, \lambda_0)} \leq \mathbb{P}[Y > c].$$

For $0 \leq c \leq \lambda_0^{-1}$, $\mathbb{P}[Y \geq c] = 1$ by definition, so clearly $\mathbb{P}[Z_{n,\operatorname{Re}\lambda} > c] \leq \mathbb{P}[Y \geq c]$. Combining these two statements gives $Y \succeq Z_{n,\operatorname{Re}\lambda}$ for $n \geq 1$ and λ such that $\operatorname{Re}\lambda \geq \lambda_0$. This implies by Lemma 8.2 that

$$p((n\operatorname{Re}\lambda R(\lambda n))^n x) \leq \mathbb{E}[q_{\lceil Z_{n,\operatorname{Re}\lambda} \rceil}(x)] \leq \mathbb{E}[q_{\lceil Y \rceil}(x)] =: q(x)$$

By the countable convexity of \mathcal{N} q is continuous and in \mathcal{N} , which proves the second statement of the theorem.

The strengthening to the case where the semigroup is of type $(M, \omega)^*$ is obvious, as it is sufficient to consider just one semi-norm $q \in \mathcal{N}$ for every $p \in \mathcal{N}$. \square

6 Generation results

The goal of this section is to prove a Hille-Yosida result for locally equi-continuous semigroups. First, we start with a basic generation result for the semigroup generated by a continuous linear operator.

Lemma 6.1. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. Suppose we have some continuous and linear operator G . Then G generates a semigroup that is strongly continuous and locally equi-continuous by defining*

$$S(t)x := \sum_{k \geq 0} \frac{t^k G^k x}{k!}. \quad (6.1)$$

Proof. As in the proof of Lemma 4.5, the fact that G is continuous implies that there exists some constant $c_G > 0$ such that for every $p \in \mathcal{N}$, there exists some $q \in \mathcal{N}$ such that $p(Gx) \leq c_G q(x)$ for all $x \in X$. Use this method to construct for a given $p \in \mathcal{N}$ an increasing sequence of semi-norms $q_n \in \mathcal{N}$, $q_0 := p$, such that $q_n(Gx) \leq c_G q_{n+1}(x)$ for every $n \geq 0$ and $x \in X$.

We consider the finite sum approximations of the sum in (6.1),

$$\begin{aligned} p\left(\sum_{k=0}^n \frac{t^k G^k x}{k!}\right) &\leq \sum_{k=0}^n \frac{t^k}{k!} p(G^k x) \\ &\leq \sum_{k \geq 0} \frac{t^k}{k!} p(G^k x) \\ &\leq e^{tc_G} \sum_{k \geq 0} \frac{(c_G t)^k}{k!} e^{-tc_G} q_k(x) \\ &\leq e^{tc_G} q_t(x), \end{aligned} \quad (6.2)$$

where

$$q_t(x) := \sum_{k \geq 0} \frac{(c_G t)^k}{k!} e^{-tc_G} q_k(x)$$

is a continuous semi-norm by the countable convexity of \mathcal{N} . The semi-norm is independent of n , which means that the sequence of sums is a Cauchy sequence. The sequential completeness of (X, τ) shows that the sequence converges. Furthermore, equation (6.2) also shows that $S(t)$ is continuous. By stochastic domination of Poisson random variables, Lemmas 8.2 and 8.3, it follows that for $t \leq T$, we have that

$$\sup_{t \leq T} e^{-tc_G} p(S(t)x) \leq q_T(x).$$

To prove strong continuity, it suffices to check that $\lim_{t \downarrow} S(t)x = x$ for every $x \in X$ by Lemma 5.1. So again consider $p \in \mathcal{N}$, we see

$$p(S(t)x - x) = p\left(\sum_{k \geq 0} \frac{t^k G^k x}{k!} e^{-t} - x\right) \leq \sum_{k \geq 0} \frac{t^k}{k!} e^{-t} p(G^k x - x).$$

Note that the expression for $k = 0$ is 0. Therefore, the dominated convergence theorem that the limit is 0 as $t \downarrow 0$. \square

In the proof of the Hille-Yosida theorem on Banach spaces, the semigroup is constructed as the limit of semigroups generated by continuous linear operators, the Yosida approximants. In the locally convex context, we need to take special care of equi-continuity of the approximating semi-groups.

Suppose that we have a operator $(A, \mathcal{D}(A))$, and we would like to generate a semigroup e^{tA} . Then we will use the next lemma, by choosing $G(\lambda) = R(\lambda, A)$.

Lemma 6.2. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. Suppose that for every $\lambda > 0$, we have a continuous linear operator $G(\lambda)$ such that for every $p \in \mathcal{N}$ there exists $q \in \mathcal{N}$ having the property that*

$$\sup_{\lambda \geq 1} \sup_n p((\lambda n G(\lambda n))^n) \leq q(x) \quad (6.3)$$

for all $x \in X$. Then the semigroups $\{S_n(t)\}_{t \geq 0, n \geq 1}$ generated by $n^2 G(n) - n\mathbb{1}$ are jointly locally equi-continuous. If we have

$$\sup_{\lambda > 0} \sup_n p((\lambda n G(\lambda n))^n) \leq q(x)$$

then the semigroups $\{S_n(t)\}_{t \geq 0, n \geq 1}$ are jointly equi-continuous.

Proof. We prove the first statement, the second statement should then be clear. First of all, equation 6.3 implies that

$$\sup_k \sup_{\lambda \in \{\frac{n}{k} \mid n \geq k\}} p((\lambda k G(\lambda k))^k x) \leq q(x)$$

for all x , which in turn can be rewritten to

$$\sup_n \sup_{k \leq n} p((n G(n))^k x) \leq q(x) \quad (6.4)$$

for all $x \in X$.

We use equation (6.4) to show that the semigroups generated by $n^2 G(n) - n\mathbb{1}$ are jointly locally equi-continuous. We see that

$$S_n(t)x := \sum_{k \geq 0} \frac{(nt)^k (n G(n))^k x}{k!} e^{-nt},$$

which intuitively corresponds to taking the expectation of $(n G(n))^k x$ under the law of a Poisson random variable with parameter nt . We exploit this point of view, to show equi-continuity of the family $\{S_n(t)\}_{t \leq T, n \geq 1}$ for some arbitrary fixed time $T \geq 0$.

For $\mu \geq 0$, let the random variable Z_μ have a *Poisson*(μ) distribution and for $t \geq 0$ and $n \geq 1$ let $B_{n,t} := \lceil \frac{Z_{nt}}{n} \rceil$. This correspond to cutting the Poisson distribution into parts: 0 is mapped to 0, and the values $\{ln+k\}_{k=1}^n$ are mapped to $l+1$. Fix a semi-norm $p \in \mathcal{N}$, and use equation (6.4), to construct an increasing sequence of semi-norms in \mathcal{N} : $q_0 = p, q_1, \dots$ such that every pair

q_l, q_{l+1} satisfies the relation in (6.4). As a consequence, we obtain that

$$\begin{aligned}
& p(S_n(t)) \\
& \leq p \left(\sum_{k \geq 0} \frac{(nt)^k (nG(n))^k x}{k!} e^{-nt} \right) \\
& \leq p(x) e^{-nt} + \sum_{l \geq 0} \sum_{k=1}^n \frac{(nt)^{nl+k}}{(nl+k)!} e^{-nt} p((nG(n))^{nl+k} x) \\
& \leq q_0(x) e^{-nt} + \sum_{l \geq 0} \sum_{k=1}^n \frac{(nt)^{nl+k}}{(nl+k)!} e^{-nt} q_{l+1}(x) \\
& = \mathbb{P}[B_{n,t} = 0] q_0(x) + \sum_{l \geq 0} \mathbb{P}[B_{n,t} = l+1] q_{l+1}(x) \\
& = \mathbb{E} [q_{B_{n,t}}(x)].
\end{aligned} \tag{6.5}$$

We see that, as in the proof of the second property in theorem 5.5, we are done if we can find a random variable Y that stochastically dominates all $B_{n,t}$ for $n \geq 1$ and $t \leq T$.

We calculate the tail probabilities of $B_{n,t}$ in the case that $t > 0$. If $t = 0$, all tail probabilities are 0. By definition,

$$\mathbb{P}[B_{n,t} > k] = \mathbb{P}[Z_{nt} > nk] = \mathbb{P} \left[\frac{1}{n} Z_{nt} > k \right].$$

As Z_{nt} is *Poisson*(nt) distributed, we can write it as $Z_{nt} = \sum_{i=1}^n X_i$ where $\{X_i\}_{i \geq 0}$ are independent and *Poisson*(t) distributed. This implies that we can apply Chernoff's bound to $\frac{1}{n} Z_{nt}$, see Proposition 8.4. First of all, for all $\theta \in \mathbb{R}$, we have $\mathbb{E} [e^{\theta X}] = \exp\{t(e^\theta - 1)\}$. Evaluating the infimum in Chernoff's bound for $k \geq \lceil T \rceil$, $T \geq t$ yields

$$\mathbb{P}[B_{n,t} > k] = \mathbb{P} \left[\frac{1}{n} Z_{nt} > k \right] < e^{-n(k \log \frac{k}{t} - k + t)}.$$

Define the function

$$\begin{aligned}
& \phi : [\lceil T \rceil, \infty) \times (0, T] \rightarrow [0, \infty) \\
& (a, b) \mapsto a \log \frac{a}{b} - a + b,
\end{aligned}$$

so that for $k \geq \lceil T \rceil$, $T \geq t$, we have $\mathbb{P}[B_{n,t} > k] < e^{-n\phi(k,t)}$.

We define a new random variable Y on $\{n \in \mathbb{N} \mid n \geq \lceil T \rceil\}$ by putting $\mathbb{P}[Y = \lceil T \rceil] = 1 - e^{-\phi(\lceil T \rceil, T)}$, and for $k \geq \lceil T \rceil$: $\mathbb{P}[Y > k] = e^{-\phi(k, T)}$, or stated equivalently $\mathbb{P}[Y = k+1] = e^{-\phi(k, T)} - e^{-\phi(k+1, T)}$. In other words, we construct Y so that the tail variables agree with $e^{-\phi(k, T)}$.

For $k < \lceil T \rceil$, we have by definition that $\mathbb{P}[Y > k] \geq \mathbb{P}[B_{n,t} > k]$ as the probability on the left is 1. For $k \geq \lceil T \rceil$, an elementary computation shows that for $t \leq T$ the function $\phi_k(t) := \phi(k, t)$ is decreasing in t . This implies that

$$\mathbb{P}[B_{n,t} > k] \leq e^{-n\phi(k,t)} \leq e^{-\phi(k,t)} \leq e^{-\phi(k, T)} = \mathbb{P}[Y > k].$$

In other words, $Y \succeq B_{n,t}$ for all $n \geq 1$ and $0 < t \leq T$. For the remaining cases, where $n \geq 1$ and $t = 0$, the result is clear as $B_{n,t} = 0$ with probability 1. By lemma 8.2 and equation 6.5, we obtain that

$$p(S_n(t)) \leq \mathbb{E}[q_{B_{n,t}}(x)] \leq \mathbb{E}[q_Y(x)] =: q(x).$$

For the second inequality, we use that Y stochastically dominates $X_{n,t}$ for all $n \geq 1$ and $t \leq T$. The semi-norm $q(x)$ is in \mathcal{N} by the countable convexity of \mathcal{N} . We conclude that the family $\{S_n(t)\}_{t \leq T, n \geq 1}$ is equi-continuous. \square

Lemma 6.3. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. Let $(A, \mathcal{D}(A))$ be a closed, densely defined operator such that there exists an $\omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and such that for every $\lambda_0 > \omega$ and semi-norm $p \in \mathcal{N}$, there is a continuous semi-norm q such that $\sup_{\lambda \geq \lambda_0} p((\lambda - \omega)R(\lambda)x) \leq q(x)$ for every $x \in X$. As $\lambda \rightarrow \infty$, we have*

(a) $\lambda R(\lambda)x \rightarrow x$ for every $x \in X$

(b) $\lambda AR(\lambda)x = \lambda R(\lambda)Ax \rightarrow Ax$ for every $x \in \mathcal{D}(A)$.

The lemma can be proven as in the Banach space case. We have now developed enough machinery to prove a Hille-Yosida type theorem.

Theorem 6.4. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. For a linear operator $(A, \mathcal{D}(A))$ on (X, τ) , the following are equivalent.*

(a) $(A, \mathcal{D}(A))$ generates a strongly continuous semigroup of type (M, ω) .

(b) $(A, \mathcal{D}(A))$ is closed, densely defined and there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and for every semi-norm $p \in \mathcal{N}$ and $\lambda_0 > \omega$ there exists a semi-norm $q \in \mathcal{N}$ such that for all $x \in X$ one has

$$\sup_{n \geq 1} \sup_{\lambda \geq \lambda_0} p((n(\lambda - \omega)R(n\lambda))^n x) \leq Mq(x).$$

(c) $(A, \mathcal{D}(A))$ is closed, densely defined and there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that for every $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > \omega$, one has $\lambda \in \rho(A)$ and for every semi-norm $p \in \mathcal{N}$ and $\lambda_0 > \omega$ there exists a semi-norm $q \in \mathcal{N}$ such that for all $x \in X$ and $n \in \mathbb{N}$

$$\sup_{n \geq 1} \sup_{\operatorname{Re} \lambda \geq \lambda_0} p((n(\operatorname{Re} \lambda - \omega)R(n\lambda))^n x) \leq Mq(x).$$

By a simplification of the arguments, we can also give a necessary and sufficient condition for the generation of a quasi equi-continuous semigroup.

Theorem 6.5. *Let $(X, \tau, \|\cdot\|)$ be a space of type B. For a linear operator $(A, \mathcal{D}(A))$ on (X, τ) , the following are equivalent.*

(a) $(A, \mathcal{D}(A))$ generates a strongly continuous semigroup of type $(M, \omega)^*$.

(b) $(A, \mathcal{D}(A))$ is closed, densely defined and there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and for every semi-norm $p \in \mathcal{N}$ there exists a semi-norm $q \in \mathcal{N}$ such that for all $x \in X$ one has

$$\sup_{n \geq 1} \sup_{\lambda > \omega} p((n(\lambda - \omega)R(n\lambda))^n x) \leq Mq(x).$$

(c) $(A, \mathcal{D}(A))$ is closed, densely defined and there exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that for every $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re} \lambda > \omega$, one has $\lambda \in \rho(A)$ and for every semi-norm $p \in \mathcal{N}$, there exists a semi-norm $q \in \mathcal{N}$ such that for all $x \in X$ and $n \in \mathbb{N}$

$$\sup_{n \geq 1} \sup_{\operatorname{Re} \lambda > \omega} p((n(\operatorname{Re} \lambda - \omega)R(n\lambda))^n x) \leq Mq(x).$$

Proof of Theorem 6.4. (a) to (c) is the content of Proposition 5.4 and Theorem 5.5 and (c) to (b) is clear. So we need to prove (b) to (a).

First note that we can always assume that $\omega = 0$ by a suitable rescaling. We first prove the result for $\omega = 0$ and $M = 1$. We follow the lines of the proof of the Hille-Yosida theorem, theorem II.3.5, in Engel and Nagel [6] for Banach spaces.

Define for every $n \in \mathbb{N} \setminus \{0\}$ the Yosida approximants

$$A_n := nAR(n) = n^2R(n) - n\mathbb{1}.$$

These operators commute and satisfy the conditions in Lemmas 6.1 and 6.2, and thus generate jointly locally equi-continuous strongly continuous commuting semigroups $t \mapsto T_n(t)$ of type $(1, 0)$. We show that this sequence of semigroups has a τ limit.

Let $x \in \mathcal{D}(A)$ and $t \geq 0$, the Fundamental theorem of calculus applied to $s \mapsto T_m(t-s)T_n(s)x$ for $s \leq t$, yields

$$\begin{aligned} T_n(t)x - T_m(t)x &= \int_0^t T_m(t-s)(A_n - A_m)T_n(s)x \, ds \\ &= \int_0^t T_m(t-s)T_n(s)(A_nx - A_mx) \, ds. \end{aligned}$$

By Lemma 6.2 and Lemma 6.3 (b), we obtain that for every semi-norm $p \in \mathcal{N}$ there exists $q \in \mathcal{N}$ such that

$$p(T_n(t)x - T_m(t)x) \leq tq(A_nx - A_mx), \quad (6.6)$$

hence, $n \mapsto T_n(s)x$ is a τ -Cauchy sequence uniformly for $s \leq t$. Define the point-wise limit of this sequence by $T(s)x := \lim_n T_n(s)x$. This directly yields that the family $\{T(s)\}_{s \leq t}$ is equi-continuous, because it is contained in the closure of an equi-continuous set of operators by Lemma 6.2 and Proposition 32.4 in Treves [21]. Consequentially, this shows that $\{T(t)\}_{t \geq 0}$ is a locally equi-continuous set of operators of type $(1, 0)$.

The fact that $\{T(t)\}_{t \geq 0}$ is a semigroup follows from the fact that it is the point-wise limit of the semigroups $\{T_n(t)\}_{t \geq 0}$. We show that it is strongly continuous by using Lemma 5.1 (c) to (a). Let $p \in \mathcal{N}$ and $x \in \mathcal{D}(A)$, then for every n :

$$p(T(t)x - x) \leq p(T(t)x - T_n(t)x) + p(T_n(t)x - x).$$

As $p(T(t)x - T_n(t)x) \rightarrow 0$, uniformly for $t \leq 1$, we can first choose n large to make the first term on the right hand side small, and then t small, to make the second term on the right hand side small.

We still need to prove that the semigroup $\{T(t)\}_{t \geq 0}$ has generator $(A, \mathcal{D}(A))$. Denote with $(B, \mathcal{D}(B))$ the generator of $\{T(t)\}_{t \geq 0}$. For $x \in \mathcal{D}(A)$, we have for a continuous semi-norm p that

$$\begin{aligned} & p\left(\frac{T(t)x - x}{t} - Ax\right) \\ & \leq p\left(\frac{T(t)x - T_n(t)x}{t}\right) + p\left(\frac{T_n(t)x - x}{t} - A_n x\right) + p(A_n x - Ax). \end{aligned}$$

As a consequence of equation (6.6), we obtain

$$\begin{aligned} & p\left(\frac{T(t)x - x}{t} - Ax\right) \\ & \leq q(Ax - A_n x) + p\left(\frac{T_n(t)x - x}{t} - A_n x\right) + p(A_n x - Ax) \end{aligned}$$

such that by first choosing n large and then t small, we see that $x \in \mathcal{D}(B)$ and $Bx = Ax$. In other words, $(B, \mathcal{D}(B))$ extends $(A, \mathcal{D}(A))$.

For $\lambda > 0$, we know that $\lambda \in \rho(A)$, so $\lambda - A : \mathcal{D}(A) \rightarrow X$ is bijective. As B generates a semigroup of type $(1, 0)$, we also have that $\lambda - B : \mathcal{D}(B) \rightarrow X$ is bijective. But B extends A , which implies that $(A, \mathcal{D}(A)) = (B, \mathcal{D}(B))$.

Now we prove the general with bound $(M, 0)$ from the $(1, 0)$ case as in the proof of Theorem II.3.8 in Engel and Nagel [6]. The strategy is to define a norm on X that is equivalent to $\|\cdot\|$ for which the semigroup that we want to construct is $(1, 0)$ bounded. First define

$$\|x\|_\mu := \sup_{n \geq 0} \|\mu^n R(\mu)^n x\|$$

and then define $\|x\| := \sup_{\mu > 0} \|x\|_\mu$. This norm has the property that $\|x\| \leq \|x\| \leq M \|x\|$ and $\|\lambda R(\lambda)\| \leq 1$ for every $\lambda > 0$. Use this norm to define a new set of continuous semi-norms as in definition 4.1 by setting

$$\mathcal{N}^* := \{p \mid p \text{ is a } \tau \text{ continuous semi-norm such that } p(\cdot) \leq \|\cdot\|\}.$$

As a consequence of $\|\lambda R(\lambda)\| \leq 1$ and the τ continuity of $\lambda R(\lambda)$, that for every $p \in \mathcal{N}^*$ there exists $q \in \mathcal{N}^*$ such that $p(\lambda R(\lambda)x) \leq q(x)$ for every $x \in X$. Likewise, we obtain for every $\lambda_0 > 0$ that for every $p \in \mathcal{N}^*$ there exists $q \in \mathcal{N}^*$ such that

$$\sup_{\lambda \geq \lambda_0} \sup_{n \geq 1} p((nR(n))^n x) \leq q(x).$$

This means that we can use the first part of the proof to construct a strongly continuous locally equi-continuous semigroup $\{T(t)\}_{t \geq 0}$ that has bound $(1, 0)$ with respect to \mathcal{N}^* .

Let $T \geq 0$. Pick a semi-norm $p \in \mathcal{N}$. It follows that $p \in \mathcal{N}^*$, so there exists a $q \in \mathcal{N}^*$ such that $\sup_{t \leq T} p(T(t)x) \leq q(x)$ for all $x \in X$.

Because $\|\cdot\| \leq M \|\cdot\|$, it follows that \mathcal{N}^* is a subset of $M\mathcal{N}$ which implies that $\hat{q} := \frac{1}{M}q \in \mathcal{N}$. We obtain $\sup_{t \leq T} p(T(t)x) \leq M\hat{q}(x)$ for all $x \in X$.

In other words, A generates a strongly continuous and locally equi-continuous semigroup $\{T(t)\}_{t \geq 0}$ of type $(M, 0)$. \square

7 Application: Markov semigroups on a complete separable metric space

Let (E, d) be a complete separable metric space. We will define the *strict* topology on $(C_b(E))$ which is suitable for the study of the transition semigroup of a Markov process on E . The strict topology interpolates between the sup norm topology and the topology of uniform convergence on compact sets.

After defining the topology, we will consider the transition semigroup of a Markov process on a locally compact space and the transition semigroup corresponding to a Markov process constructed via the martingale problem.

7.1 Definition and basic properties of the strict topology

For every compact set $K \subseteq E$, define the semi-norm $p_K(f) := \sup_{x \in K} |f(x)|$. The *compact open* topology κ on $C_b(E)$ is generated by the semi-norms $\{p_K : K \text{ compact}\}$. Now define semi-norms in the following way. Pick a non-negative sequence a_n in \mathbb{R} such that $a_n \rightarrow 0$. Also pick an arbitrary sequence of compact sets $K_n \subseteq E$. Define

$$p_{(K_n), (a_n)}(f) := \sup_n a_n p_{K_n}(f). \quad (7.1)$$

The *strict* topology β defined on $C_b(E)$ is generated by the semi-norms

$$\{p_{(K_n), (a_n)} \mid K_n \text{ compact}, 0 < a_n \rightarrow 0\},$$

see Theorem 3.1.1 in Wiweger [25] and Theorem 2.4 in Sentilles [18]. Note that in the latter paper, the topology introduced here is called the substrict topology. However, he shows in Theorem 9.1 that the strict and the substrict topology coincide when the underlying space E is Polish.

Obviously, $C_b(E)$ can also be equipped with the sup norm topology.

Theorem 7.1. *The locally convex space $(C_b(E), \beta)$ together with the sup norm is of type A and B.*

This result follows from the following theorem, which summarises some known results about $(C_b(E), \beta)$. Property (f) indicates why this topology is suitable for the study of Markov processes. For property (g), we say that $A \subseteq C_b(E)$ *separates points* if for every $x, y \in E$, $x \neq y$, there is $f \in A$ such that $f(x) \neq f(y)$. Furthermore, we say that A *vanishes nowhere* if for every $x \in E$, there is $f \in A$ such that $f(x) \neq 0$.

Theorem 7.2. *$(C_b(E), \beta)$ satisfies the following properties.*

- (a) $(C_b(E), \beta)$ is complete.
- (b) $(C_b(E), \beta)$ is a Mackey space.
- (c) $(C_b(E), \beta)$ is a sequential space. In other words, the topology is determined by sequential convergence. Consequentially, $C_b(E)' = C_b(E)^+$.
- (d) A set $B \subseteq C_b(E)$ is bounded in β if and only if it is bounded in the norm topology. Furthermore, restricted to bounded sets B , it holds that $\beta|_B = \kappa|_B$.

(e) A sequence $f_n \subseteq C_b(E)$ converges to $f \in C_b(E)$ with respect to β if and only if (a) and (b) hold:

(i) $\sup_n \|f_n\| < \infty$,

(ii) $f_n \rightarrow f$ with respect to κ , in other words, if for every compact set $K \subseteq E$: $\lim_n \sup_{x \in K} |f_n(x) - f(x)| = 0$.

(f) The dual of $(C_b(E), \beta)$ is the space of Radon measures of finite total variation norm.

(g) An algebra $A \subseteq C_b(E)$ that separates points and vanishes nowhere is dense in $(C_b(E), \beta)$.

Proof. (a) and (f) follow from Theorem 9.1 in Sentilles [18], (b) follows from Theorems 5.7 and 9.1 in [18], (c) from Corollary 8.3 [18]. (d) follows from 2.2.1 and the Corollary of 2.4.1 in Wiweger [25] and (e) follows from Theorem 2.3.1 in Wiweger [25]. (g) is the Stone-Weierstrass theorem found for example as Theorem 10 in Fremlin, Garling and Haydon [8]. \square

In this situation, the set \mathcal{N} contains all semi-norms of the type given in equation (7.1) such that $\sup_n a_n \leq 1$.

7.2 Semigroups on a locally compact space

In the case that (E, d) is a locally compact space, both $(C_b(E), \beta)$ and $(C_0(E), \|\cdot\|)$ have the space of Radon measures as a dual. As such the space of Radon measures carries two weak topologies. The first one is the one that probabilists call the *weak* topology, i.e. $\sigma(\mathcal{M}(E), C_b(E))$, and the second is the weaker *vague* topology, i.e. $\sigma(\mathcal{M}(E), C_0(E))$. As such, we expect that if a semigroup is strongly continuous on $(C_b(E), \beta)$ it is strongly continuous on $(C_0(E), \|\cdot\|)$, as long as the semigroup maps $C_0(E)$ into itself. We denote with $\mathcal{P}(E) \subseteq \mathcal{M}(E)$ the probability measures.

Theorem 7.3. *Let $\{S(t)\}_{t \geq 0}$ be a strongly continuous semigroup on $(C_b(E), \beta)$ such that $S(t)C_0(E) \subseteq C_0(E)$ for every $t \geq 0$. The restriction of the semigroup to $C_0(E)$, denoted by $\{\tilde{S}(t)\}_{t \geq 0}$ is $\|\cdot\|$ strongly continuous.*

Conversely, suppose that we have a strongly continuous semigroup $\{\tilde{S}(t)\}_{t \geq 0}$ on $(C_0(E), \|\cdot\|)$ such that $\tilde{S}'(t)\mathcal{P}(E) \subseteq \mathcal{P}(E)$. Then the semigroup can be extended uniquely to a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ on $(C_b(E), \beta)$.

Proof. We start with the first statement. For a given time $t \geq 0$, the operator $S(t)$ is β to β continuous. Therefore, it maps bounded sets into bounded sets, which implies that $S(t)$ is norm continuous. Therefore, also the restriction $\tilde{S}(t)$ of $S(t)$ to $C_0(E)$ is norm continuous.

As $\{S(t)\}_{t \geq 0}$ is $(C_b(E), \beta)$ is strongly continuous, it is also weakly continuous, in other words, for every radon measure μ and $f \in C_b(E)$, we have that $t \mapsto \langle S(t)f, \mu \rangle$ is continuous. This holds in particular for $f \in C_0(E)$. Theorem I.5.8 in Engel and Nagel [6] yields that the semigroup $\{\tilde{S}(t)\}_{t \geq 0}$ is strongly continuous on $(C_0(E), \|\cdot\|)$.

We proceed with the proof of the second statement. Our first goal is to extend $\tilde{S}(t)$ to $C_b(E)$. By Theorem 7.2 (g), $C_0(E)$ is dense in $(C_b(E), \beta)$. So if we can

show that $\tilde{S}(t)$ is β to β continuous on $C_0(E)$, we can construct the extension by continuity.

In fact, we will directly show that $\{\tilde{S}(t)\}_{t \geq 0}$ is locally β equi-continuous.

First of all, by the completeness of $(C_b(E), \beta)$, the fact that $C_0(E)$ is dense in $(C_b(E), \beta)$ and 21.4.(5) in Köthe [11], we have $(C_0(E), \beta)' = (C_b(E), \beta)' = \mathcal{M}(E)$ and the equi-continuous sets in $\mathcal{M}(E)$ with respect to $(C_0(E), \beta)$ and $(C_b(E), \beta)$ coincide.

It follows by 39.3.(4) in Köthe [12] that $\{\tilde{S}(t)\}_{t \geq 0}$ is locally β equi-continuous if for every $T \geq 0$ and β equi-continuous set $K \subseteq \mathcal{M}(E)$ we have that

$$SK := \{\tilde{S}'(t)\mu \mid t \leq T, \mu \in K\}$$

is β equi-continuous. By Theorem 6.1 (c) in Sentilles [18], it is sufficient to prove this result for β equi-continuous sets K consisting of non-negative measures in $\mathcal{M}(E)$. Furthermore, we can restrict to weakly closed K , as the weak closure of an equi-continuous set is β equi-continuous.

So let K be an arbitrary weakly closed β equi-continuous subset of the non-negative Radon measures. We show that SK is weakly compact, as this will imply β equi-continuity by Proposition 3.1.

By Theorem 8.9.4 in Bogachev [3], we obtain that the weak topology on the positive cone in $\mathcal{M}(E)$ is metrisable. So, we only need to show sequential weak compactness of SK . Let ν_n be a sequence in SK . Clearly, $\nu_n = \tilde{S}'(t_n)\mu_n$ for some sequence $\mu_n \in K$ and $t_n \leq T$. As K is β equi-continuous, it is weakly compact by the Bourbaki-Alaoglu theorem, so without loss of generality, we restrict to a weakly converging subsequence $\mu_n \in K$ with limit $\mu \in K$ and $t_n \rightarrow t$, for some $t \leq T$.

Now there are two possibilities, either $\mu = 0$, or $\mu \neq 0$.

In the first case, we obtain directly that $\nu_n = \tilde{S}'(t_n)\mu_n \rightarrow 0$ weakly. In this case it clearly holds that $0 \in SK$, so we have found a weakly converging subsequence in SK . In the second case, one can show that

$$\hat{\mu}_n := \frac{\mu_n}{\langle \mathbb{1}, \mu_n \rangle} \rightarrow \frac{\mu}{\langle \mathbb{1}, \mu \rangle} =: \hat{\mu}$$

weakly, and therefore vaguely. By the computation in lemma 3.2, more specifically equation (3.1), we obtain that $\tilde{S}'(t_n)\hat{\mu}_n \rightarrow \tilde{S}'(t)\hat{\mu}$ vaguely. By assumption, all measures involved are probability measures, so by Proposition 3.4.4 in Ethier and Kurtz [7] implies that the convergence is also in the weak topology. By an elementary computation, we infer that the result also holds without the normalising constants: $\nu_n \rightarrow \tilde{S}'(t)\mu$ weakly.

So both cases give us a weakly converging subsequence in SK .

We conclude that $\{\tilde{S}(t)\}_{t \leq T}$ is β equi-continuous. So we can extend all $\tilde{S}(t)$ by continuity to β continuous maps $S(t) : C_b(E) \rightarrow C_b(E)$. Also, we directly obtain that $\{S(t)\}_{t \geq 0}$ is locally β equi-continuous. The semigroup property of $\{S(t)\}_{t \geq 0}$ follows from the semigroup property of $\{\tilde{S}(t)\}_{t \geq 0}$. The strong continuity follows directly from Proposition 3.3. \square

7.3 A semigroup corresponding to a Markov process constructed via the martingale problem

A method that is often to construct Markov processes is via the so called martingale problem. The method was introduced in Stroock and Varadhan [19, 20],

and afterwards applied in many other situations. For more references, see Section 4.12 in Ethier and Kurtz [7]. We introduce some notation. Let $D_E(\mathbb{R}^+)$ be the Skorokhod space of càdlàg E valued paths, i.e. paths that are right continuous and have limits from the left. For background on this space see for example Section 3.5 in Ethier and Kurtz [7].

We start with some background on the martingale problem. Suppose that (E, d) is a compact space. Furthermore, suppose that we have a process X , represented by a measure $\mathbb{P} \in \mathcal{P}(D_E(\mathbb{R}^+))$, such that the transition semigroup $\{S(t)\}_{t \geq 0}$ is strongly continuous on $(C(E), \|\cdot\|)$. Using Lemma 5.2 (d), we can show that for every $f \in \mathcal{D}(A)$

$$f(X(t)) - f(X(0)) - \int_0^t Af(X(s))ds \quad (7.2)$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $\mathcal{F}_t = \sigma(X(s) \mid s \leq t)$. The martingale problem takes this probabilistic property as a starting point.

We return to the general situation of a complete separable metric space (E, d) . Solving the martingale problem is a probabilistic approach to construct a Markov process that corresponds formally to having a generator

$$(\hat{A}, \mathcal{D}(\hat{A})), \quad \hat{A} : \mathcal{D}(\hat{A}) \subseteq C_b(E) \rightarrow C_b(E)$$

in the form of equation (7.2). We give a rigorous definition.

Definition 7.4. Given a probability measure $\nu \in \mathcal{P}(E)$, we say that the process X with measure $\mathbb{P}_\nu \in \mathcal{P}(D_E(\mathbb{R}^+))$ solves the martingale problem for $(\hat{A}, \mathcal{D}(\hat{A}))$, where $\mathcal{D}(\hat{A})$ separates points, with initial measure ν , written as \mathbb{P}_ν solves (\hat{A}, ν) if the following are satisfied:

- (a) $X(0)$ is distributed according to ν .
- (b) For every $f \in \mathcal{D}(\hat{A})$

$$f(X(t)) - f(X(0)) - \int_0^t \hat{A}f(X(s))ds$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $\mathcal{F}_t = \sigma(X(s) \mid s \leq t)$.

The goal for this section, is to show that the martingale method is related to semigroup theory on $(C_b(E), \beta)$. To do this, we need to be able to start from any initial distribution. Also, we assume some regularity for the solutions.

Condition 7.5. For every $x \in E$, there is a solution \mathbb{P}_x to the martingale problem (\hat{A}, δ_x) such that

- (a) For every compact set $K \subseteq E$, $\varepsilon > 0$ and $T > 0$ there exists a compact set $\hat{K} := \hat{K}(K, \varepsilon, T)$, $K \subseteq \hat{K}$, such that

$$\sup_{x \in K} \mathbb{P}_x \left[X(t) \in \hat{K} \text{ for all } t \leq T \right] \geq 1 - \varepsilon$$

- (b) $x \mapsto \mathbb{P}_x$ is weakly continuous.

Define the transition semigroup by $S(t)f(x) = E[f(X(t)) \mid X(0) = x]$.

Lemma 7.6. *For every $t \geq 0$, $S(t)$ is a maps $C_b(E)$ into $C_b(E)$.*

Proof. Fix $t \geq 0$. Pick a sequence x_n converging to x in E . By assumption $\mathbb{P}_{x_n} \rightarrow \mathbb{P}_x$ weakly. By Theorem 4.3.12 in Ethier and Kurtz [7], we obtain $\mathbb{P}_x[X(t) = X(t-)] = 1$, where $X(t-)$ is the left hand limit of the trajectory $s \mapsto X(s)$ at $s = t$. By Theorem 3.7.8 in Ethier and Kurtz and the fact that $\mathbb{P}_x[X(t) = X(t-)] = 1$, we obtain that $X_n(t) \rightarrow X(t)$ in distribution, or in other words, $S(t)f(x_n) \rightarrow S(t)f(x)$. \square

Lemma 7.7. *For every $t \geq 0$, the map $S(t) : (C_b(E), \beta) \rightarrow (C_b(E), \beta)$ is continuous.*

Proof. Fix $t \geq 0$. We will prove that $S(t)$ is β continuous by using Theorem 7.2 (c) and (e). Pick a sequence f_n converging to f with respect to β . It follows that $\sup_n \|f_n\| \leq \infty$, which directly implies that $\sup_n \|S(t)f_n\| < \infty$.

We also know that $f_n \rightarrow f$ uniformly on compact sets. We prove that this implies the same for $S(t)f_n$ and $S(t)f$. Fix $\varepsilon > 0$ and a compact set $K \subseteq E$, and let \hat{K} be the set introduced in Condition 7.5 (a) for $T = t$. Then we obtain that

$$\begin{aligned} & \sup_{x \in K} |S(t)f(x) - S(t)f_n(x)| \\ & \leq \sup_{x \in K} \mathbb{E}_x |f(X(t)) - f_n(X(t))| \\ & \leq \sup_{x \in K} \mathbb{E}_x \left| (f(X(t)) - f_n(X(t))) \mathbb{1}_{\{X(t) \in \hat{K}\}} + (f(X(t)) - f_n(X(t))) \mathbb{1}_{\{X(t) \in \hat{K}^c\}} \right| \\ & \leq \sup_{y \in \hat{K}} |f(y) - f_n(y)| + \sup_n \|f_n - f\| \varepsilon. \end{aligned}$$

As $n \rightarrow \infty$ this quantity is bounded by $\sup_n \|f_n - f\| \varepsilon$ as f_n converges to f uniformly on compacts. As ε was arbitrary, we are done. \square

Proposition 7.8. *$\{S(t)\}_{t \geq 0}$ is β strongly continuous.*

We give a probabilistic proof below, using the fact that weak continuity and local equi-continuity implies strong continuity. A second proof along the lines of the proof of Proposition 7.9 allows a direct proof of strong continuity.

Proof. By Condition 7.5 it is clear that the semigroup $\{S(t)\}_{t \geq 0}$ is locally equi-continuous. Therefore, Theorem 3.4 implies that we only need to check weak continuity. So let $f \in C_b(E)$ and $\mu \in \mathcal{M}(E)$. Write μ as the Hahn-Jordan decomposition: $\mu = c^+ \mu^+ - c^- \mu^-$, where $c^+, c^- \geq 0$ such that $\mu^+, \mu^- \in \mathcal{P}(E)$. We show that $t \mapsto \langle S(t)f, \mu \rangle$ is continuous, by showing that $t \mapsto \langle S(t)f, \mu^+ \rangle$ and $t \mapsto \langle S(t)f, \mu^- \rangle$ are continuous. Clearly, it suffices to do this for either of the two.

Construct the measure \mathbb{P}_{μ^+} on $D_E(\mathbb{R}^+)$ as $\int \mathbb{P}_x \mu^+(dx)$. It is clear that \mathbb{P}_{μ^+} solves the (\hat{A}, μ^+) martingale problem. It follows by Theorem 4.3.12 in Ethier and Kurtz [7] that $\mathbb{P}_{\mu^+}[X(t) = X(t-)] = 1$ for all $t > 0$, so $t \mapsto X(t)$ is continuous \mathbb{P}_{μ^+} almost surely. Fix some $t > 0$, we show that our trajectory is continuous for this specific t .

$$|\langle S(t)f, \mu^+ \rangle - \langle S(t+h)f, \mu^+ \rangle| \leq \mathbb{E}_{\mu^+} |f(X(t)) - f(X(t+h))|.$$

By the almost sure convergence of $X(t+h) \rightarrow X(t)$ as $h \rightarrow 0$, and the boundedness of f , we obtain by the dominated convergence theorem that this difference converges to 0 as $h \rightarrow 0$. As $t > 0$ was arbitrary, the trajectory is continuous for all $t > 0$. Continuity at 0 follows by the fact that all trajectories in $D_E(\mathbb{R}^+)$ are continuous at 0.

In other words, $\{S(t)\}_{t \geq 0}$ is weakly continuous, which concludes the proof by Theorem 3.4. \square

As a strongly continuous semigroup, $S(t)$ has a corresponding infinitesimal generator A .

Proposition 7.9. *A is an extension of \hat{A} .*

Proof. We show that if $f \in \mathcal{D}(\hat{A})$, then $f \in \mathcal{D}(A)$. We again use the characterisation of β convergence as given in Theorem 7.2. From this point onward, we write $g := \hat{A}f$ to ease the notation.

First, $\sup_t \left\| \frac{S(t)f - f}{t} \right\| \leq \|g\|$ as

$$\frac{S(t)f(x) - f(x)}{t} = \mathbb{E}_x \left[\frac{f(X(t)) - f(x)}{t} \right] = \mathbb{E}_x \left[\frac{1}{t} \int_0^t g(X(s)) ds \right]$$

Second, we show that we have uniform convergence of $\frac{S(t)f - f}{t}$ to g as $t \downarrow 0$ on compact sets. So pick $K \subseteq E$ compact. Now choose $\varepsilon > 0$ arbitrary, and let $\hat{K} = \hat{K}(K, \varepsilon, 1)$ as in Condition 7.5.

$$\begin{aligned} & \sup_{x \in K} \left| \frac{S(t)f(x) - f(x)}{t} - g(x) \right| \\ & \leq \sup_{x \in K} \mathbb{E}_x \left| \frac{1}{t} \int_0^t g(X(s)) ds - g(x) \right| \\ & \leq \sup_{x \in K} \mathbb{E}_x \mathbb{1}_{\{X(s) \in \hat{K} \text{ for } s \leq 1\}} \left| \frac{1}{t} \int_0^t g(X(s)) ds - g(x) \right| \\ & \quad + \sup_{x \in K} \mathbb{E}_x \mathbb{1}_{\{X(s) \notin \hat{K} \text{ for } s \leq 1\}} \left| \frac{1}{t} \int_0^t g(X(s)) ds - g(x) \right| \\ & \leq \sup_{x \in K} \mathbb{E}_x \mathbb{1}_{\{X(s) \in \hat{K} \text{ for } s \leq 1\}} \left| \frac{1}{t} \int_0^t g(X(s)) ds - g(x) \right| + 2\varepsilon \|g\| \end{aligned} \quad (7.3)$$

Thus, we need to work on the term on the last line.

The function g restricted to the compact set \hat{K} is uniformly continuous. So let $\eta > 0$, chosen smaller than ε , be such that if $d(x, y) < \eta$, $x, y \in \hat{K}$, then $|g(x) - g(y)| \leq \varepsilon$.

By Lemma 4.5.17 in Ethier and Kurtz, the set $\{\mathbb{P}_x \mid x \in K\}$ is a weakly compact set in $\mathcal{P}(D_E(\mathbb{R}^+))$. So by Theorem 3.7.2 in Ethier and Kurtz, we obtain that there exists a $\delta = \delta(\eta) > 0$ such that

$$\sup_{x \in K} \mathbb{P}_x \left[y \in D_E(\mathbb{R}^+) \mid \sup_{s \leq \delta} d(y(0), y(s)) < \eta \right] > 1 - \eta > 1 - \varepsilon.$$

Denote $S_\delta := \{y \in D_E(\mathbb{R}^+) \mid \sup_{s \leq \delta} d(y(0), y(s)) < \eta\}$, so that we can summarise the equation as $\sup_{x \in K} \mathbb{P}_x[S_\delta] > 1 - \varepsilon$.

We reconsider the term that remained in equation (7.3).

$$\begin{aligned} & \sup_{x \in K} \mathbb{E}_x \mathbb{1}_{\{X(s) \in \hat{K} \text{ for } s \leq 1\}} \left| \frac{1}{t} \int_0^t g(X(s)) - g(x) ds \right| + 2\varepsilon \|g\| \\ & \leq \sup_{x \in K} \mathbb{E}_x \mathbb{1}_{\{X(s) \in \hat{K} \text{ for } s \leq 1\} \cap S_\delta} \left| \frac{1}{t} \int_0^t g(X(s)) - g(x) ds \right| + 4\varepsilon \|g\| \end{aligned}$$

On the set $\{X(s) \in \hat{K} \text{ for } s \leq 1\} \cap S_\delta$, we know that $d(x(s), x) \leq \eta$, so that by the uniform continuity of g on \hat{K} , we obtain $|g(X(s)) - g(x)| \leq \varepsilon$. Hence:

$$\sup_{t \leq 1 \wedge \delta(\eta)} \sup_{x \in K} \left| \frac{S(t)f(x) - f(x)}{t} - g(x) \right| \leq \varepsilon + 4\varepsilon \|g\|.$$

As $\varepsilon > 0$ was arbitrary, it follows that $f \in \mathcal{D}(A)$ and $Af = g = \hat{A}f$. \square

8 Appendix: Stochastic domination and the Chernoff bound

In this appendix, we state the definition of a basic stochastic domination and give a number of useful results. For more details on stochastic domination, see for example [16, Section IV.1].

Definition 8.1. Suppose that we have two random variables η_1 and η_2 taking values on \mathbb{R} .

We say that η_1 stochastically dominates η_2 , denoted by $\eta_1 \succeq \eta_2$ if for every $r \in \mathbb{R}$ we have $\mathbb{P}[\eta_1 \geq r] \geq \mathbb{P}[\eta_2 \geq r]$.

Lemma 8.2. For two random variables η_1, η_2 on \mathbb{R} , we have that $\eta_1 \succeq \eta_2$ if and only if for every bounded and increasing function ϕ , we have $\mathbb{E}[\phi(\eta_1)] \geq \mathbb{E}[\phi(\eta_2)]$.

We say that a random variable η is Poisson(γ) distributed, $\gamma \geq 0$, denoted by $\eta \sim \text{Poisson}(\gamma)$ if $\mathbb{P}[\eta = k] = \frac{\gamma^k}{k!} e^{-\gamma}$.

Lemma 8.3. If $\eta_1 \sim \text{Poisson}(\gamma_1)$ and $\eta_2 \sim \text{Poisson}(\gamma_2)$ and $\gamma_1 \geq \gamma_2$, then $\eta_1 \succeq \eta_2$.

Using the theory of couplings [16, Section IV.2], a proof follows directly from the fact that if $\gamma_1 \geq \gamma_2$, then η_1 is in distribution equal to $\eta_2 + \zeta$, where $\zeta \sim \text{Poisson}(\gamma_1 - \gamma_2)$.

Now we turn to a tool that is useful in the context of stochastic domination, which was introduced by Chernoff [4].

Proposition 8.4. Let X be a random variable on \mathbb{R} for which there exists $\theta_0 > 0$, such that for $\theta < \theta_0$, the Laplace transform $\mathbb{E}[e^{\theta X}]$ exists. Let $\{X_i\}_{i \geq 1}$ be independent and distributed as X . Then for $c \geq \mathbb{E}[X]$, we have

$$\mathbb{P} \left[\frac{1}{n} \sum_{i=1}^n X_i > c \right] < \exp \left\{ -n \inf_{0 < \theta < \theta_0} \{c\theta - \log \mathbb{E}[e^{\theta X}]\} \right\}.$$

We give a proof for completeness.

Proof. For all $0 < \theta < \theta_0$, we have

$$\begin{aligned} \mathbb{P} \left[\frac{1}{n} \sum_{i=1}^n X_i > c \right] &= \mathbb{P} \left[e^{\theta \sum_{i=1}^n X_i} > e^{n\theta c} \right] \\ &< \exp \left\{ - \left(n\theta c - \log \mathbb{E} \left[e^{\theta \sum_{i=1}^n X_i} \right] \right) \right\}, \end{aligned}$$

where we used Markov's inequality in line 2. As the X_i are independent $\log \mathbb{E} [e^{\theta \sum_{i=1}^n X_i}] = n \log \mathbb{E} [e^{\theta X}]$, which yields the final result. \square

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